Equivalent Generic Forms for Metric Fields Yielded by Relativistic Positioning Systems

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Abstract. Relativistic positioning systems provide tensors represented in $\{\ell\ell\ell\ell\}$-frames (ℓ for light). We show, in particular, that any Lorentzian metric field given in such a frame is equivalent to a generic metric field defined by four positive functions.

Key words: Causal classes; Emission coordinates; Flags; Formal integrability; General relativity; Geometrical equivalence; Jet manifolds; Metric fields; Multi-flags; Pfaff systems; Relativistic positioning systems

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1 Introduction

Current positioning systems, such as GPS, GLONASS, Galileo, Beidou, or IRNSS, are not true relativistic positioning systems, as is now well accepted. Indeed, the data they provide, such as, for instance, the time stamps broadcast by the satellites of these constellations and collected by the receiver devices of the user, cannot be sent or used directly without prior algorithmic corrections. The latter are due, in particular, to relativistic effects, undergone, for instance, by the on-board atomic clocks, and incorporated in the realtime computations (via the so-called Kalman filters) to obtain the "correct" spacetime positions. The relativistic effects taken into account in these processes [1] are mainly the gravitational frequency shifts, the first and second-order relativistic Doppler shifts, and the Sagnac effect (even if the latter is not always considered a "true" relativistic effect by some authors). Other corrections are included and based on models such as those for signal transmissions through the ionosphere or for the Earth's geoid. But, the latter are not really at the heart of the designs of these positioning systems, unlike the relativistic effects.

To circumvent or avoid such fundamental root defects, new designs have been investigated and are thought of as providing true relativistic positioning systems (RPS). They are based on new protocols of spacetime positioning primarily devised, to our knowledge, by B. Coll, J.J. Ferrando, J.A. Morales and A. Tarantola [5, 6, 8, 9], initially in the case of two-dimensional spacetime and in four-dimensional spacetime with the SYPOR protocol for instance [11]. Moreover, E. Capolongo, M.L. Ruggiero, and A. Tartaglia recently evaluated such protocols for constructing the Earth's worldline with pulsars as celestial beacons [18, 19, 20, 21].

The system of coordinates generated by RPS are the so-called *emission coordinates* given by the time stamps broadcast by the satellites, and then, their associated $\{\ell\ell\ell\ell\}$-frames are made of four future light-like basis vectors tangent to the light-like paths of the signals. As a result, the Lorentzian metric fields g are represented in these $\{\ell\ell\ell\ell\}$-frames by matrices such as

$$g \equiv \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} \\ g_{21} & 0 & g_{23} & g_{24} \\ g_{31} & g_{32} & 0 & g_{34} \\ g_{41} & g_{42} & g_{43} & 0 \end{pmatrix}, \tag{1}$$

where $g_{ij} = g_{ji} \neq 0$ if $i \neq j$ $(i, j = 1, \dots, 4)$ and $\mathrm{sgn}(g_{ij}) = -\varepsilon$ whenever the signature of g is 2ε. B. Coll and J.M. Pozo have made an extensive study of the algebraic properties of this class of metrics [10]. They have shown, in particular, that in the general case the terms g_{ij} are not factorized, *i.e.*, no set Λ of 4 nonvanishing functions ν_i exists such that, for instance, $g_{ij} \equiv \nu_i \nu_j$ for all i, $j = 1, \dots, 4$ such that $i \neq j$.

Besides, we know that Sylvester's law of inertia gives a reduction of any quadratic form to a sum of squares, and thus, it reduces the initial number $n(n+1)/2$ of components of any n-dimensional symmetric metric to at most n nonvanishing components on the diagonal. Unfortunately, in the present case, starting with a causal $\{\ell\ell\ell\ell\}$ class for the representation of the metric field g, this causal class is necessarily lost in Sylvester's process of reduction since nonvanishing diagonal components cannot be ascribed to square norms of light-like basis vectors. Nevertheless, Sylvester's law of inertia suggests that only n functions should be needed to completely define g, regardless of its causal class of representation. However, if $n \leqslant 4$, we show that there always exists a set Λ and a metric $\tilde{g}$ which is $\{\ell\ell\ell\ell\}$-equivalent to g, in a meaning to be specified in the sequel, with factorized components. This equivalence is obtained from a change of local $\{\ell\ell\ell\ell\}$-frame related to a local change of emission coordinates.

2 The equivalent generic metric field

Let $\mathcal{M}$ be a smooth connected n-dimensional pseudo-Riemannian manifold endowed with a Lorentzian metric g represented as in (1) in a given $\{\ell\ell\ell\ell\}$-frame defined on an emission coordinates chart $(U, \tau^1, \dots, \tau^n)$ where the open $U \subset \mathcal{M}$. We denote by ∂_i the partial derivative with respect to the i-th emission coordinate τ^i of τ. Then, the present paper is devoted to the proof of the following result:

Theorem 2.1. *If $n \leqslant 4$, there always exists a smooth local diffeomorphism f of which the Jacobian matrix is orthogonal and n smooth positive functions ν_i, both defined on an open neighborhood $V \subset U$ of any given point of U, such that for all $\tau \equiv (\tau^1, \dots, \tau^n) \in V$ the relations*

$$\tilde{g}_{ij} \equiv \sum_{r,\, s=1}^{n} g_{rs}(f)(\partial_i f^r)(\partial_j f^s) = \epsilon_{ij}\, \nu_i\, \nu_j\,, \qquad i, j = 1, \dots, n\,, \tag{2}$$

*hold with $\epsilon_{ij} = \mathrm{sgn}(g_{ij}) = \mathrm{sgn}(\tilde{g}_{ij})$ whenever $i \neq j$ and $\epsilon_{ij} = 0$ otherwise. Then, we say that the "generic" metric $\tilde{g}$ is $\{\ell\ell\ell\ell\}$-*equivalent *to g (through f).*

Note that the non-diagonal terms of g are not vanishing. Furthermore, if $n \leqslant 3$, the result is trivial: take the identity map for f and the functions ν_i are unique. Moreover, if $n = 2$, we can make a separation of variables in g_{12} such that each function ν_i $(i = 1, 2)$ depends on only one emission coordinate (because any two-dimensional Riemann manifold is conformally flat) [8, 9]. In cases of dimension greater than 4, some constraints on the definition of g must be imposed.

The proof of this theorem presented below is made in the framework of the smooth category rather than the analytic category, which is the standard situation for the application of the Cartan-Kähler theorem. We do not use either this theorem or Cartan's test for involutivity. Hence, neither the computations of the codimensions of the polar spaces associated to the integral elements of certain flags nor the evaluations of their Kähler-regularities or regularities are performed [4]. The main reason is due to the non-standard way we "transform" a given set of algebraic equations defined in a jet bundle and associated to a system of PDEs to an associated Pfaff system of contact 1-forms. We just make a little step aside in the definition of this "transformation" with strong advantages in the proof as a result, as there appear to be some unnoticed forms of indetermination in the definition of the associated Pfaff system of a system of PDEs.

Proof. Let π_n be the trivial fibration $\pi_n : \mathcal{M}^2 \equiv \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$, corresponding to the projection onto the first factor. We denote by $J_k(\pi_n)$ the fiber bundle of jets of order $k \geqslant 0$ of the local smooth sections of π_n. In particular, we have $J_0(\pi_n) \equiv \mathcal{M}^2$ with local coordinates $(\tau, \psi) \equiv (\tau^1, \ldots, \tau^n, \psi^1, \ldots, \psi^n)$. Furthermore, let

$$\psi_1 \equiv (\tau^1, \ldots, \tau^n, \psi^1, \ldots, \psi^n, \psi^1_1, \psi^1_2, \ldots, \psi^i_j, \ldots, \psi^n_{n-1}, \psi^n_n)$$

be a local system of coordinates on $J_1(\pi_n)$. We denote also by $\Pi_k(\pi_n) \subset J_k(\pi_n)$ the set of invertible elements of $J_k(\pi_n)$, *i.e.*, the set of k-jets of local smooth diffeomorphisms on $\mathcal{M}$. $\Pi_k(\pi_n)$ is a groupoid with source map $\alpha_k : \Pi_k(\pi_n) \longrightarrow \mathcal{M}$ where $\mathcal{M}$ is the first factor of $\mathcal{M}^2$ and the target map $\beta_k : \Pi_k(\pi_n) \longrightarrow \mathcal{M}$ where we project onto the second factor. Also, we denote by $\mathit{\Pi}_k(\pi_n)$ the presheaf of germs of local smooth α_k-sections of $\Pi_k(\pi_n)$. Then, we consider any solution of the system of PDEs (2) as a sub-manifold of $\Pi_1(\pi_n)$ transversal to the α_k-fibers and defined from the following system $\mathcal{R}_1$ of equations on the presheaf $\mathit{\Pi}_1(\pi_n)$:

$$\sum_{r,\,s=1}^{n} g_{rs}(\psi)\, \psi^r_i\, \psi^s_j - \epsilon_{ij}\, \nu_i\, \nu_j = 0\,, \qquad i, j = 1, \ldots, n\,, \tag{3}$$

where $\nu_i > 0$.

Then, we denote also by $\hat{g}_{ij}$ the terms such that

$$\hat{g}_{ij} \equiv \sum_{r,\,s=1}^{n} g_{rs}(\psi)\, \psi^r_i\, \psi^s_j\,, \qquad i, j = 1, \ldots, n\,.$$

Hence, $\mathcal{R}_1$ is also the following set of algebraic equations:

$$\mathcal{R}_1 : \quad \begin{cases} \hat{g}_{ii} = 0\,, \\ \hat{g}_{ij} = \epsilon_{ij}\, \nu_i\, \nu_j\,, \qquad i \neq j = 1, \ldots, n\,. \end{cases}$$

We deduce easily for all distinct indices i, j and k that $\epsilon_{ij}\, \epsilon_{jk}\, \hat{g}_{ij}\, \hat{g}_{jk} = (\nu_j)^2\, \hat{g}_{ik}\, \epsilon_{ik}$. Then, we must have

$$\mathcal{R}_1 : \quad \begin{cases} \hat{g}_{ii} = 0\,, \\ |\hat{g}_{ij}\, \hat{g}_{jk}| = (\nu_j)^2\, |\hat{g}_{ik}|\,, \qquad \text{for all } i,\ j \text{ and } k \text{ distinct in } \{1, \ldots, n\}. \end{cases} \tag{4}$$

In particular, if $n = 4$ in (4), then, apart from the set of equations $\hat{g}_{ii} = 0$, the second set of equations are necessarily satisfied unless the two following deduced equations are not:

$$|\hat{g}_{12}\, \hat{g}_{34}| = |\hat{g}_{13}\, \hat{g}_{24}|\,, \qquad |\hat{g}_{13}\, \hat{g}_{24}| = |\hat{g}_{14}\, \hat{g}_{23}|\,.$$

Therefore, if $n = 4$, the system $\mathcal{R}_1$ reduces to the following set of PDEs:

$$\mathcal{R}_1 : \quad \begin{cases} \hat{g}_{ii} = 0\,, \qquad i = 1, \ldots, 4\,, \\ |\hat{g}_{12}\, \hat{g}_{34}| = |\hat{g}_{13}\, \hat{g}_{24}| \neq 0\,, \\ |\hat{g}_{13}\, \hat{g}_{24}| = |\hat{g}_{14}\, \hat{g}_{23}| \neq 0\,. \end{cases}$$

Rewriting this system of PDEs without the absolute values, we obtain

$$\mathcal{R}_1 : \quad \begin{cases} \mathcal{F}_i(\psi_1) \equiv \sum_{r,s=1}^{4} g_{rs}(\psi)\, \psi^r_i\, \psi^s_i = 0\,, \qquad i = 1, \ldots, 4\,, \\ \mathcal{F}_5(\psi_1) \equiv \sum_{i,j,k,h=1}^{4} g^{\epsilon}_{ijkh}(\psi)\, \psi^i_1\, \psi^j_2\, \psi^k_3\, \psi^h_4 = 0\,, \\ \mathcal{F}_6(\psi_1) \equiv \sum_{i,j,k,h=1}^{4} g^{\epsilon'}_{ijhk}(\psi)\, \psi^i_1\, \psi^j_2\, \psi^k_3\, \psi^h_4 = 0\,, \end{cases} \tag{5}$$

where

$$g^{\epsilon}_{ijkh} \equiv g_{ij}\, g_{kh} - \epsilon\, g_{ik}\, g_{jh}\,, \qquad \epsilon = \pm 1\,, \quad i, j, k, h = 1, \ldots, 4\,.$$

Before going further, we must know if there exist solutions to the system of homogeneous polynomial equations (5) in the variables ψ^i_j whenever ψ is fixed. First, we denote by ϕ_i the linearly independent column vectors such that $\phi_i \equiv (\psi^j_i)$. From the first four equations $\mathcal{F}_i(\psi_1) = 0$ $(i = 1, \ldots, ;, 4)$, the former must be light-like vectors which still exist since g is Lorentzian. Second, the last two functions can be rewritten as $\mathcal{F}_5(\psi_1) = g(\phi_1, \phi_5)$ and $\mathcal{F}_6(\psi_1) = g(\phi_1, \phi_6)$ where $\phi_5 \equiv \hat{g}_{34}\, \phi_2 - \epsilon\, \hat{g}_{24}\, \phi_3$ and $\phi_6 \equiv \hat{g}_{34}\, \phi_2 - \epsilon' \hat{g}_{23}\, \phi_4$. Therefore, the nonvanishing vectors ϕ_5 and ϕ_6 are collinear to ϕ_1 or time-like ($\hat{g}_{ij} \neq 0$ if $i \neq j$). However, because the four vectors ϕ_i $(i = 1, \ldots, 4)$ are linearly independent, then ϕ_5 and ϕ_6 must be time-like. Hence, the signs of their norms $g(\phi_5, \phi_5)$ and $g(\phi_6, \phi_6)$ are equal to the sign of the signature 2ε of g, *i.e.*, we have

$$\mathrm{sgn}(\hat{g}_{34}\, \hat{g}_{24}\, \hat{g}_{23}) = -\epsilon\varepsilon = -\epsilon'\varepsilon\,.$$

Thus, in particular, we must have $\epsilon' = \epsilon$ in the system (5). Besides, ϵ is arbitrary, and then, from now and throughout, we set also $\epsilon = \varepsilon$. As a result, we have solutions to the system (5) if and only if [1]

$$\hat{g}_{34}\, \hat{g}_{24}\, \hat{g}_{23} < 0\,. \tag{6}$$

Next, we consider the expression $\hat{g}_{34}\, \hat{g}_{24}\, \hat{g}_{23}$ as a quadratic form Q with respect to ϕ_2. We obtain $Q(\phi_2, \phi_2) \equiv \sum_{i,j=1}^4 Q_{ij}\, \phi^i_2\, \phi^j_2$ where $Q_{ij} = \hat{g}_{34} \sum_{r,s=1}^4 g_{ir}\, g_{js}\, \phi^r_4\, \phi^s_3$, and then, the inequality (6) is always satisfied if Q is not a positive elliptic form. For, it suffices that one of the diagonal terms Q_{ii} to be non-positive since, in this case, it implies the existence of basis vectors of non-positive norms with respect to Q.[2] Thus, we impose, in particular, the condition $Q_{11} \leqslant 0$. Proceeding in the same way, the term Q_{11} is still considered as a quadratic form R with respect to ϕ_3. And again, we have $Q_{11} \leqslant 0$ if R is not a positive elliptic form. This condition is always satisfied since $R_{11} \equiv g_{11} \sum_{i,j=1}^4 g_{1i}\, g_{1j}\, \phi^i_4\, \phi^j_4 = 0$. Hence, there always exist real solutions to the system (5) whatever are the source τ and the target ψ.

Additionally, from the inequality (6) and the 'continuity of roots' property [22, p.363], we deduce that, given a point ψ, there always exists a maximal open subset $U_\psi \subset \mathcal{M}$ of ψ such that this set of solutions S_ψ is always an open smooth manifold of *constant* dimension at least 10 on U_ψ. As a result, U_ψ is also necessarily closed, but then, because $\mathcal{M}$ is connected, we deduce that $\dim S_\psi = m$ is a constant on $\mathcal{M}$. Moreover, $\alpha_1 \times \beta_1$ is a surmersion on $\mathcal{M}^2$, and thus, the latter has no critical points in $\mathcal{R}_1$. Therefore, we obtain that $m = 10$ [15, see Lemma 1, p.11].

It follows that the restrictions to $\mathcal{R}_1$ of the source and target maps are surmersions, and then, the system $\mathcal{R}_1$ is, respectively, *formally integrable* (as a system of local diffeomorphisms defined on the whole of $\mathcal{M}$), and *homogeneous* (transitive diffeomorphisms from opens to any other opens in $\mathcal{M}$). And then, $\mathcal{R}_1$ is a differentiable manifold such that $\dim \mathcal{R}_1 = 18$.

Next, we consider the following canonical contact structure S_0 of width n (*i.e.*, n-flag [13, 17]) and length 1 on $\Pi_1(\pi_n)$ generated by the set $\{\omega^1, \omega^2, \ldots, \omega^n\}$ of contact 1-forms $\omega^i \in T^*J_1(\pi_n)$

[1] Note that this inequality illustrates the first form of the Tarski-Seidenberg theorem [2, 12].

[2] We can use also the Coll-Morales rules [7, see § III] generalizing more effectively the Jacobi, Gundelfinger and Frobenius rules with the notion of *causal sequence* $(i_1, i_2, i_3) \equiv (\mathrm{sgn}(\triangle_1), \mathrm{sgn}(\triangle_2), \delta\, \mathrm{sgn}(\triangle_3))$ where the $\triangle_k$'s are the first three *leading principal minors* of Q of order k and δ is the *determinant index*. In the present case, the causal sequence should differ from the causal sequence $(1, 1, 1)$.

such that

$$
S_0 : \quad \begin{cases} \omega^1 = & d\psi^1 - \sum_{i=1}^n \psi_i^1 \, d\tau^i \, , \\ \omega^2 = & d\psi^2 - \sum_{j=1}^n \psi_j^2 \, d\tau^j \, , \\ \ldots = & \ldots\ldots\ldots\ldots\ldots\ldots \, , \\ \omega^n = & d\psi^n - \sum_{k=1}^n \psi_k^n \, d\tau^k \, . \end{cases} \tag{7}
$$

Obviously, the *terminal system* S_1 of S_0 is vanishing. Then, we complement the set of contact 1-forms generating S_0 with another set of 1-forms ω_j^i on $\Pi_1(\pi_n)$ defined by the relations:

$$
\omega_j^i \equiv d\psi_j^i - \sum_{k=1}^n z_{jk}^i(\psi_1) \, d\tau^k \, , \qquad\qquad i, j = 1, \ldots, n \, , \tag{8}
$$

where any given set of functions $z_{jk}^i(\psi_1) \in C^\infty(\Pi_1(\pi_n))$ must satisfy

$$
z_{jk}^i(\psi_1) = z_{kj}^i(\psi_1) \, , \qquad\qquad D_k z_{jh}^i(\psi_1) = D_h z_{jk}^i(\psi_1) \, , \qquad\qquad i, j, k, h = 1, \ldots, n \, , \tag{9}
$$

where D_k is the formal differentiation with respect to τ^k defined by the formula

$$
D_k \equiv \frac{\partial}{\partial \tau^k} + \sum_{i=1}^n \psi_k^i \frac{\partial}{\partial \psi^i} + \sum_{i,j=1}^n z_{jk}^i(\psi_1) \frac{\partial}{\partial \psi_j^i} \, , \qquad\qquad k = 1, \ldots, n \, .
$$

From this definition and for any smooth function $\mathcal{F}$ defined on $J_1(\pi_n)$ we find that the commutator $[D_k, \, D_h]$ satisfies the relation

$$
[D_k, \, D_h](\mathcal{F}) = \sum_{i,j=1}^n \left(D_k z_{jh}^i - D_h z_{jk}^i \right) \frac{\partial \mathcal{F}}{\partial \psi_j^i} \, . \tag{10}
$$

Then, we denote by $T_0(z) \supseteq S_0$ this new contact structure generated by the contact 1-forms ω^i and the 1-forms ω_j^i $(i, j = 1, \ldots, n)$. In particular, from (10) and the relation $d^2\omega = 0$ for any smooth p-forms ω in $\Lambda T^* J_1(\pi_n)$, we deduce also that the *Martinet structure tensor* $\delta \equiv d$ mod $T_0(z)$ is such that $\delta^2 = 0$ [14].

Moreover, from relations (7) and (8), we obtain:

$$
\begin{cases} d\omega^i = \sum_{k=1}^n d\tau^k \wedge \omega_k^i \, , \\ d\omega_k^j = \sum_{h,r,s=1}^n \left(\frac{\partial z_{kh}^j}{\partial \psi_s^r} \right) d\tau^h \wedge \omega_s^r + \sum_{h,r=1}^n \left(\frac{\partial z_{kh}^j}{\partial \psi^r} \right) d\tau^h \wedge \omega^r \, , \end{cases}
$$

and then, $T_0(z)$ satisfies the Frobenius conditions (equivalent to $\delta^2 = 0$) and is an integrable Pfaff system on $\Pi_1(\pi_n)$.

Next, we consider $\mathcal{R}_1$ as a presheaf $\mathcal{I}_1$ of ideals locally finitely generated by the functions $\mathcal{F}_i$ $(i = 1, \ldots, 6)$ defined on $\Pi_k(\pi_4)$ and we assume that any manifold on which this presheaf vanishes, *i.e.*, the sub-manifold defined from a solution, is an integral sub-manifold of a $T_0(z)$ in $\Pi_k(\pi_4)$. We denote by $V_1(z)$ the foliation of all of these integral sub-manifolds. This latter version conforms better with the classical concepts of integral manifolds and differs from the approach of PDEs translated in terms of presheafs of Pfaff systems of contact 1-forms satisfying the Frobenius conditions (see for instance [3]).

As a consequence, denoting by $\mathcal{J}_1(z)$ the presheaf of differential ideals generated by $T_0(z)$ on $J_1(\pi_4)$, we say that $\mathcal{R}_1$ is integrable on $\mathcal{M}$ if there exists a sub-manifold of solutions $V_1(z) \subseteq \Pi_1(\pi_4)$ and a nonvanishing presheaf $\mathcal{J}_1(z)$ such that $\mathcal{I}_1 \subseteq \mathcal{J}_1(z)$ on $V_1(z)$.

In other words, if a set of functions z^i_{jk} exists satisfying the latter condition, a smooth local diffeomorphism f of $\mathcal{M}$ is a solution of $\mathcal{R}_1$ if and only if

$$\begin{cases} \mathcal{F}_i(j_1(f)) = \iota_0\,, & i = 1, \dots, 6\,, \\ f^*(\omega^i) = 0\,, & f^*(\omega^j_k) = 0\,, \qquad i, j, k = 1, \dots, 4\,, \end{cases}$$

where ι_0 is the zero function on $\mathcal{M}$ and $j_1(f)$ is the first prolongation of f; and thus a local section of $\Pi_1(\pi_4)$. Hence, from (8), we obtain that

$$\begin{cases} df^i = \sum_{k=1}^4 (\partial_k f^i)\, d\tau^k\,, & i = 1, \dots, 4\,, \\ df^j_k = \sum_{h=1}^4 z^j_{kh}(j_1(f))\, d\tau^h\,, & j, k = 1, \dots, 4\,. \end{cases}$$

And then, from the second order of derivation and from the successive prolongations, all of the derivatives of f are functionals of the derivatives of f of order less than or equal to one. As a result, a Taylor expansion for f can be deduced with Taylor coefficients defined from the Taylor coefficients of f of order less than or equal to one only. Thus, we obtain a formal Taylor expansion for f which can be convergent on a suitable relatively compact open neighborhood U_τ of any point $\tau \in \mathcal{M}$ if some Lipschitzian conditions on the functions z^j_{kh} are satisfied on $(\alpha_1)^{-1}(U_\tau) \cap V_1(z)$; justifying the definition of integrability given above for $\mathcal{R}_1$.

To satisfy the condition $\mathcal{I}_1 \subseteq \mathcal{J}_1(z)$ on a manifold $V_1(z)$, we must have $d\mathcal{F}_i \equiv 0 \mod T_0(z)$ for all of the indices $i = 1, \dots, 6$ on $V_1(z)$. We obtain the following system $\mathcal{S}(z)$ of 24 linear equations with 24 unknowns z^i_{jk}:

$$\delta\mathcal{F}_i = 0 \Longrightarrow \sum_{r,s=1}^4 \left(\left(\sum_{k=1}^4 (\partial_k g_{rs})(\psi)\, \psi^k_h \right) \psi^r_i\, \psi^s_i + g_{rs}(\psi)\, \psi^r_i\, z^s_{ih} \right) = 0\,,$$

$$\delta\mathcal{F}_5 = 0 \Longrightarrow \sum_{i,j,k,h,r=1}^4 (\partial_r g^\varepsilon_{ijkh})(\psi)\, \psi^i_1\, \psi^j_2\, \psi^k_3\, \psi^h_4\, \psi^r_s$$
$$+ \sum_{i,j,k,h=1}^4 g^\varepsilon_{ijkh} \left\{ \psi^i_1\, \psi^j_2\, \psi^k_3\, z^h_{4s} + \psi^i_1\, \psi^j_2\, z^k_{3s}\, \psi^h_4 + \psi^i_1\, z^j_{2s}\, \psi^k_3\, \psi^h_4 + z^i_{1s}\, \psi^j_2\, \psi^k_3\, \psi^h_4 \right\} = 0\,,$$

$$\delta\mathcal{F}_6 = 0 \Longrightarrow \sum_{i,j,k,h,r=1}^4 (\partial_r g^\varepsilon_{ijhk})(\psi)\, \psi^i_1\, \psi^j_2\, \psi^k_3\, \psi^h_4\, \psi^r_s$$
$$+ \sum_{i,j,k,h=1}^4 g^\varepsilon_{ijhk} \left\{ \psi^i_1\, \psi^j_2\, \psi^k_3\, z^h_{4s} + \psi^i_1\, \psi^j_2\, z^k_{3s}\, \psi^h_4 + \psi^i_1\, z^j_{2s}\, \psi^k_3\, \psi^h_4 + z^i_{1s}\, \psi^j_2\, \psi^k_3\, \psi^h_4 \right\} = 0\,.$$

Note that if $n > 4$ we have more equations than unknowns, and then not all metric fields g are admissible to satisfy the conditions of the theorem. Now, setting for all of the functions z^i_{jk} the relations

$$z^i_{jk}(\psi_1) = \psi^i_j \sum_{h=1}^4 \psi^h_k\, z_h(\psi)\,, \tag{11}$$

where the functions z_k depend on ψ, we find that the unique solution of $\mathcal{S}(z)$ is the set of functions z^i_{jk} such that

$$z_k(\psi) = -\frac{1}{8} \sum_{i,j=1}^4 g^{ij}(\psi)\, (\partial_k g_{ij})(\psi) \equiv -\frac{1}{4} \sum_{i=1}^4 \Gamma^i_{ik}(\psi)\,, \tag{12}$$

where the Γ^i_{jk} are the Christoffel symbols of g. Then, it remains to see that the conditions (9) are satisfied. For, we must have the relations

$$\sum_{h=1}^{4}\left(z^i_{jr}(\psi_1)\,\psi^h_k - z^i_{jk}(\psi_1)\,\psi^h_r\right) z_h(\psi) = 0\,,$$

which are, actually, verified with the functions z^i_{jk} given by the relations (11) with (12). Moreover, because no algebraic constraints exist on ψ_1, apart from those obtained from the vanishing of the functions $\mathcal{F}_i$ which are elements of $\mathcal{I}_1 \subseteq \mathcal{J}_1(z)$, then the manifold $V_1(z)$ is the whole of the open manifold $\Pi_1(\pi_4)$. Furthermore, the 1-forms ω^k and ω^i_j are the so-called *basic 1-forms* [16] associated with any *complete transversally parallelizable* foliation. Lastly, at any given point τ, the finite system of equations (5) in the variables ψ_1 always have solutions, and the set of positive functions ν_i is not unique. ∎

Besides, we note that $\mathcal{R}_1$ is not a Lie groupoid because if g is $\{\ell\ell\ell\ell\}$-equivalent to $\epsilon_{ij}\,\nu_i\,\nu_j$ and $\epsilon_{ij}\,\tilde{\nu}_i\,\tilde{\nu}_j$ through, respectively, the diffeomorphisms f and $\tilde{f}$, then, there may not always be four positive functions $\hat{\nu}_i$ such that g would be $\{\ell\ell\ell\ell\}$-equivalent to $\epsilon_{ij}\,\hat{\nu}_i\,\hat{\nu}_j$ through $f\circ\tilde{f}$ or $\tilde{f}\circ f$. Nevertheless, we have an associated *principal groupoid* regarded as the graph of the $\{\ell\ell\ell\ell\}$-equivalence relation, and then the equivalence class $[g]$ of the given metric g is a source fiber in this groupoid. Moreover, if $\mathcal{M}$ is time oriented, *i.e.*, there exists a complete (future time-like) vector field ξ on $\mathcal{M}$, then, because $\mathcal{R}_1$ is also a differentiable α_1-fiber bundle, we also have in the smooth category the following:

Theorem 2.2. *If $n = 4$, then, given a Lorentzian metric g on $\mathcal{M}$ assumed to be time oriented, connected and simply connected, then, there exists only one smooth diffeomorphism $f^i(\tau) \equiv \psi^i$ being a solution of $\mathcal{R}_1$ of which the Jacobian matrix is an element of $SO(4)$; and, as a result, there is a unique set of four positive functions ν_i.*

Proof. Let $\psi_0 \equiv (\tau,\,\psi)$ be any point in $\mathcal{M}^2$ and a matrix $\Psi \equiv (\psi^i_j) \in \alpha_1^{-1}(\tau)\times\beta_1^{-1}(\psi) \equiv \mathcal{R}_1^{\psi_0}$. Then, in particular, we have $\det\Psi \neq 0$, and from the precedent proof we have also $\dim\mathcal{R}_1^{\psi_0} = 10$. Let ψ_0 be a fixed point, then the coefficients ψ^i_j of Ψ satisfy a system consisting of the six homogeneous equations (5). If, moreover, the four column vectors $\phi_k \equiv (\psi^i_k)$ $(k = 1,\ldots,4)$ are orthogonal each to the others, then, additionally, Ψ verifies a system consisting of six multivariate quadratic equations $Q_i(\Psi) = 0$ $(i = 1,\ldots,6)$ (the six scalar products of the four column vectors ϕ_k). Hence, let r_{ψ_0} be the smooth map such that $r_{\psi_0} : \Psi \in \mathcal{R}_1^{\psi_0} \longrightarrow (Q_1(\Psi),\ldots,Q_6(\Psi)) \in \mathbb{R}^6$, then, we can show that $\ker r_{\psi_0}$ is a nonempty four dimensional manifold [15, see Lemma 1, p.11]. Indeed, the tangent map Tr_{ψ_0} is regular in $\mathcal{R}_1^{\psi_0}$ because the coefficients of Tr_{ψ_0} are linear with respect to Ψ, and then, if $\det Tr_{\psi_0} = 0$, we would have the four vectors ϕ_k not linearly independent, which is not possible from the relation $\det\Psi \neq 0$. In addition, because the twelve polynomials Q_i and $\mathcal{F}_j$ are homogeneous, then the four vectors ϕ_k can be normalized, and thus, $\Psi \in SO(4)$. It follows that 1) $S^{\psi_0} \equiv SO(4)\cap\mathcal{R}_1^{\psi_0}$ is not empty, and 2) S^{ψ_0} is a real semialgebraic set consisting of sixteen homogeneous multivariate polynomial equations of even degrees and one inequation. Consequently, because there are as many algebraic equations than unknowns, we obtain a nonempty *finite* [2, 12, § 2.3] set $s(\psi_0)$ of real roots $\Psi \in SO(4)$ which are solutions of the system (5). Moreover, from the 'continuity of roots' property, the continuity of g on $\mathcal{M}$ and the connexity of $\mathcal{M}^2$, we deduce that $|s(\psi_0)|$ is constant over $\mathcal{M}^2$; And then, the set $\cup_{\psi_0\in\mathcal{M}^2}S^{\psi_0}$ is a covering of $\mathcal{M}^2$ which is universal because $\mathcal{M}^2$ is simply connected. Therefore, there is only one preimage of ψ_0 under $\alpha\times\beta$ in S^{ψ_0}. ∎

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